Proof of the contrapositive

It is a direct proof but we start with the contrapositive because

$$P \implies Q \text{ is equivalent to } \neg(Q) \implies \neg(P).$$

Why do we prove the contrapositive of the implication instead of the original implication?

What is the difference between *proof of the contrapositive* and *proof by contradiction*?

Example 1

Definition: An integer x is called *even* (respectively *odd*) if there is another integer k for which x = 2k (respectively 2k+1).

Definition: Two integers are said to have the *same parity* if they are both odd or both even.

Theorem: If x and y are two integers for which x+y is even, then x and y have the same parity.

Example 2

Definition: An integer n is called a *perfect square* if there is another integer k such that $n = k^2$.

For example, 121 is a perfect square since $121 = 11^2$.

Theorem If n is a positive integer such that $n \text{ mod}(4)$ is 2 or 3, then n is not a perfect square.

Where could such a theorem be useful?

Exercises

1. Prove n is even if n^3 is even.

2. Let $n, a, b \in \mathbb{Z}$. If $n \nmid ab$, then $n \nmid a$ and $n \nmid b$.

Principle of Mathematical Induction :

For each natural number n, let $P(n)$ be a statement. We like to demonstrate that $P(n)$ is true for all $n \in \mathbb{N}$.

To show that $P(n)$ holds for **all** natural numbers n, it suffices to establish the following:

I. Base case: Show that $P(0)$ is true.
(If $n \geq 1$, then we should start from $P(1)$.)

II. Induction step:

(i) **Assume** that $P(k)$ holds for an **arbitrary** $k \in \mathbb{N}$.
This step is "Induction hypothesis".

(ii) Show that "$P(k)$ is true" (a hypothesis) **implies** "$P(k+1)$ is also true" (a conclusion).

III. P(n) is true for all $n \in \mathbb{N}$

Proof by induction

This is sometimes referred to as the domino effect. Once one of the dominoes topples it causes the rest to topple as well.

$P(0) \implies P(1)$, $P(1) \implies P(2)$, $P(2) \implies P(3)$, $\cdots$. $P(k) \implies P(k+1)$.

All Horses Have the Same Color

I. Base case: One horse

The case with just one horse is trivial. If there is only one horse in the “group”, then clearly all horses in that group have the same color.

II. Induction step:

Assume that n horses always are the same color. Let us consider a group consisting of n+1 horses.

First, exclude the last horse and look only at the first n horses; all these are the same color since n horses always are the same color.

Likewise, exclude the first horse and look only at the last n horses. These too, must also be of the same color.

Therefore, the first horse in the group is of the same color as the horses in the middle, who in turn are of the same color as the last horse. Hence the first horse, middle horses, and last horse are all of the same color, and we have proven that:

If n horses have the same color, then n+1 horses will also have the same color.

III. Conclusion:

We already saw in the base case that the rule (“all horses have the same color”) was valid for n=1.

The inductive step showed that since the rule is valid for n=1, it must also be valid for n=2, which in turn implies that the rule is valid for n=3 and so on.

Thus in any group of horses, all horses must be the same color.

Do all horses indeed have the same color? Is the proof wrong or is the Principle of Mathematical Induction not valid?